Ising-Model Analysis of the Developer-Neighbor (D-N) Problem.[1]

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1. Introduction.

We report here a theoretical treatment of the well-known D-N, or Developer-Neighbor problem.[2] [4] [6] Our analysis provides deep understanding of it and determines its optimal resolution.

We assert that the principal features of the D-N problem are displayed in the Residential-Commercial (R-C) system at equilibrium. We analyze the Residential-Commercial equilibrium by use of the Ising model:[7] an area, such as that of the City of Tucson, is divided into sites[8] on a square grid; each site can be in one of two states, Commercial (C) or Residential (R);[10] [11] a site interacts with its nearest neighbors; the strengths of the site-site interactions (RR, RC, CR, CC) and the total number of sites (the size of the system) are variable parameters; the values of these or equivalent parameters can be determined or set, and when given can be used to calculate properties of the system.

We can ask questions of this model: (a) what fraction of the area (number of sites) are in the Commercial or the Residential state for a particular set of parameter values (interaction strengths and total number of sites)? (b) do sites of the same state cluster, and if so, what is the origin of the clustering? (c) are there ranges of parameter values for which the system is specially sensitive (catastrophes)? (d) how can we optimize the system?

Our application of the Ising model to the D-N problem is analogous to application of the model to the coil-helix transition of a protein. The Ising model implies, for a city as for a protein, the possibility of a phase transition and extreme sensitivity of the R-C equilibrium to changes in system conditions (the environment). We note also that a city, like a protein, is a small system, in the sense it comprises very many fewer than Avogadro's number of elements.

The discussion that follows is necessarily technical. Partly in expiation, the text is less succinct, somewhat pedagogical, and more voluminously footnoted[12] than the typical scientific communication. We suggest to the Reader that the footnotes be followed along with the text.

2. Theory[13]

In order to obtain a simple expression of the model, we treat the one-dimensional case. This can be solved by the matrix method.[14] [15] [16] Zimm and Bragg modeled in this way the phase transition between coil and helix states of a polypeptide chain.[17] For convenience,[18] we closely follow their treatment.[20]

2.1 Site-Site Interactions

Site-site interactions are usually given as energies. To attach physical meaning to this model of the Residential-Commercial equilibrium, we assert that a dollar value is equivalent to an energy.[21]

2.1.1 Reference state. We choose as reference state a Residential site inside a region of contiguous Residential sites. The reference state is the zero of energy or of dollar value. This selection of a reference state eliminates one of the four interaction-energy parameters noted above.

2.1.2 $\mathbf{i_{CC}}$ is the change in energy or dollar value[22] associated with transformation of one Residential site inside a Residential region into a Commercial site inside a Commercial region, i.e., expansion of a Commercial region by one site at the expense of a Residential region. Using the monetary picture,

$$i_{CC} = \$P + [\delta\$_{CC} - \delta\$_{RR}] \qquad \text{(Eq. 1a)}$$

where $\$P$ is the Profit/Commercial-site, $\delta\$_{CC}$ is the contribution from interactions of a Commercial site with its neighbors within a pure Commercial region, and similarly, $\delta\$_{RR}$ is the contribution from interactions of a Residential site with its neighbors within a pure Residential region. Being the sum of a profit and a difference between two numbers of comparable size, the value of i_{CC} typically is small.[23]

To simplify notation and follow Zimm and Bragg, we define the new parameter

$$s = e^{i_{CC}} \qquad \text{(Eq. 1b)}$$

$$i_{CC} = \log s$$

The parameter s is the microscopic equilibrium constant for interconversion of a Residential site and a Commercial site, each being inside their respective regions. Because i_{CC} is small, s will be close to unity.

2.1.3 $\mathbf{i_{RC}}$ is the change in energy or dollar value when initiating a region (or run) of Commercial sites adjacent to and following a region (or run) of Residential sites. Equivalently, we can picture i_{RC} as the cost of creating a *new* Residential-Commercial interface region; or as the cost of initiating a *new* Commercial region by transforming a Residential site into a Commercial site inside a Residential region. The value of i_{RC} is typically large and negative, consisting of a small favorable contribution from i_{CC} plus a large cost of initiating a new Commercial region within a previously pristine Residential region. The initiation cost is the sum of several contributions.[25] The cost need be paid only once per Commercial region, but if it is large, initiation of a new Commercial region is difficult.

To isolate the initiation cost and so to simplify the model, following Zimm and Bragg, we define the parameter σ by

$$\sigma s = e^{i_{RC}} \quad \text{(Eq. 2a)}$$

Noting that $s = e^{i_{CC}}$,

$$\log \sigma = i_{RC} - i_{CC} \quad \text{(Eq. 2b)}$$

$$= (cost\ of\ creation\ of\ R - C\ interface) \quad \text{(Eq. 2c)}$$

If the cost of creating the Residential-Commercial interface is large and negative, $\sigma \ll 1$. We use the value $\sigma = 10^{-2}$, unless stated otherwise.

2.1.4 $\mathbf{i_{CR}}$. If i_{RC} is the cost in energy or dollar value associated with initiating a new Commercial region, then we may picture i_{CR} as a similar cost associated with terminating a Commercial region. By symmetry with i_{RC}, i_{CR} can be pictured as the cost of initiating a *new* Residential region inside a Commercial region. Following Zimm and Bragg, we set i_{CR} at zero. This is mathematically sound: one can choose to lump all of the interface effect into i_{RC}.[26]

By setting i_{CR} at zero, we have eliminated a second of the four interaction energy parameters noted above. Thus only three variable parameters are sufficient to define the system: s, the microscopic Residential-Commercial equilibrium constant, σ, the initiation cost for a new Commercial region, and n, the total number of Residential and Commercial sites in the system.

2.2 Partition Function

Our aim is to calculate the partition function Z, knowledge of which allows us to evaluate the equilibrium properties of a system.[27] The classical partition function is the sum of the statistical weights of all possible arrangements of the system, in this case, all possible ways of assigning Commercial and Residential states to each site.

For the one-dimensional Ising model with the lattice (chain) wrapped and connected last element to first, one finds easily by the matrix method that

$$Z = Tr\{\mathbf{M}^n\} = \lambda_0^n + \lambda_1^n \quad \text{(Eq. 3)}$$

where n is the total number of sites, $\mathbf{M}$ is the transfer matrix,[29]

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ \sigma s & s \end{pmatrix} \quad \text{(Eq. 4)}$$

and $\lambda_0 > \lambda_1$ are the two eigenvalues of **M**, with

$$\lambda = \frac{1}{2}(1 + s \pm [(1-s)^2 + 4\sigma s]^{1/2}) \qquad \text{(Eq. 5)}$$

Note that Z is a function only of the eigenvalues λ and of n. The eigenvalues λ are functions only of s and σ. Thus, Z is a function only of the two interaction parameters, s and σ, or equivalently i_{CC} and i_{RC}, and the system size, n:

$$Z = f(s, \sigma, n) \qquad \text{(Eq. 6)}$$

2.3 Characterization of the System

Following Zimm and Bragg[17][30] and armed with our knowledge of the partition function Z, we first give, without proof and as functions of Z and the parameters s, σ and n, expressions for variables that describe an urban Residential-Commercial equilibrium system; then we estimate values for the variables and the parameters.

2.3.1 Fraction Commercial state, θ, and Residential-Commercial equilibrium constant, K_C. The fraction of sites in the Commercial state,

$$\theta = \frac{n_C}{n_C + n_R} \qquad \text{(Eq. 7a)}$$

$$= \frac{1}{n-3}\,\frac{d \log Z}{d \log s} \qquad \text{(Eq. 7b)}$$

The overall (macroscopic) equilibrium constant describing the interconversion of Commercial and Residential states,

$$K_{RC} = \frac{n_C}{n_R} = \frac{\theta}{1-\theta} \qquad \text{(Eq. 8)}$$

Fig. 1 gives plots of the function θ over the $\log_{10} n - \log_{10} s$ plane, calculated for several values of σ (0.01, 0.1, and 1) by use of Eq. 7b. The low-θ region at low s corresponds to essentially pure Residential, the high-θ region at high s, to essentially pure Commercial. The transition between regions becomes increasingly sharp with smaller σ. For large n, the Residential-Commercial transition is essentially independent of system size n and the transition is centered at $s \approx 1$.

2.3.2 Critical size, n_{crit}. There is a size for the system, $n = n_{crit}$, below which a Commercial region is unstable and not likely to exist. This is a consequence of the cost associated with initiation of a new Commercial region inside an existing Residential region. For a Commercial region to form, this large cost, given by $\log \sigma$ (Eq. 2c), must be balanced by the sum of many small gains in value, given by $n_{crit} \cdot \log s$ (Eq. 1b), for the transfer of n_{crit} sites from a Residential region to a Commercial region, giving[31]

$$n_{crit} = -\frac{\log \sigma}{\log s} \qquad \text{(Eq. 9)}$$

The value of n_{crit} is larger for smaller σ (more difficult initiation) and is smaller for larger s (larger profit for each site transferred, R $\rightarrow$ C).

There are two corollaries of Eq. 9: (a) If there is a significant initiation cost ($\sigma < 1$), Commercial sites are with high probability part of a cluster of Commercial sites (a strip mall, etc.), and if the initiation cost is large ($\sigma \ll 1$), the clusters will be large (n_{crit} large). (b) If the system size n is small, the Residential-Commercial transition occurs at a value of the equilibrium constant s significantly greater than 1 (Fig. 1): s must be sufficiently large that agglomeration of a small number of sites overcomes the cost of cluster initiation.

2.3.3 Average number of Commercial regions in the system, ν. The average number of Commercial regions

$$\nu = \frac{d \log Z}{d \log \sigma} \qquad \text{(Eq. 10)}$$

For n sufficiently large and $s \approx 1$,

$$\nu \approx \frac{n}{2}\sqrt{\sigma} \qquad \text{(Eq. 11a)}$$

and for $s > 1$,

$$\nu \approx \frac{n}{s-1}\,\sigma \qquad \text{(Eq. 11b)}$$

Fig. 2 is a contour plot of $\log_{10} \nu$ over the $\log_{10} n - \log_{10} s$ plane, calculated for $\sigma = 0.01$ by use of Eq. 10 (solid lines, contours for specified values of $\log_{10} \nu$). The dashed lines are the 0.1 and 0.9 contours for θ, limits which define the transition region between pure Residential at small s and pure Commercial at high s.

For fixed system size n and n large (a horizontal cut across the graph at $\log_{10} n > 1.5$), the number of Commercial regions ν is a maximum in the central transition region (equilibrium constant $s \approx 1$). This is expected: at high values of s, essentially all sites are Commercial and joined into only one or a small number of separate large clusters; at low values of s, there is essentially no Commercial, and thus only a few separate small clusters.

2.3.4 Average size of a Commercial region, S_C. The average number of sites per Commercial region,

(Eq. 12a)

$$S_C = n \cdot \theta / \nu$$

For n sufficiently large and $s \approx 1$,

$$S_C \approx \frac{1}{\sqrt{\sigma}} \qquad \text{(Eq. 12b)}$$

Eq. (12b) is a path to evaluation of σ.

Fig. 3 is a contour plot of $\log_{10} S_C$ over the $\log_{10} n - \log_{10} s$ plane, for $\sigma = 0.01$. For $n > 1/\sigma$, the approximation of Eq. 12b is seen to be accurate.

A theoretically correct estimate of n_{crit} is the value of S_C at the midpoint of the Residential-Commercial transition (halfway between the dashed lines corresponding to contours for θ values of 0.1 and 0.9, Fig. 3).

For n small and $\log_{10} s > 0.5$, n_{crit} estimated by Eq. 9 is in agreement with the results of the calculations shown in Fig. 3, based on Eq. 10 and 12a. As noted, Eq. 9 fails for $s \approx 1$, i.e., $\log_{10} s < 0.1$.

For n large, the cluster size at the midpoint of the transition, n_{crit}, is independent of system size n. For the parameter values used in the calculations of Fig. 3, we find in the transition region that the cluster size $S_C = \sigma^{-2} = 10$ for $n > 20$. As a consequence of constant cluster size, the number of clusters in the system, ν, must for large n increase in direct proportion to n, as is found (Fig. 2).

S_C is the correlation length. The correlation length is central to the description of phase transitions (scaling and power laws).

2.4 Estimation of Values

In order to work with the above expressions based on the partition function, we need a set of values for the parameters σ, s, and n that are self-consistent and are plausibly grounded in observation. For subsequent discussion of the Residential-Commercial equilibrium system, we need to evaluate also several functions of the parameters: θ, K_{RC}, S_C, and ν.

2.4.1 θ, $\mathbf{K}_{RC}$. The fraction of the land area of Tucson that is Commercial is ca. 20 percent,[32] giving $\theta = 0.2$ and the equilibrium constant $K_{RC} = 0.25$.

2.4.2 s. For a Residential-Commercial system in the transition region $(0.1 < \theta < 0.9)$, and with $\sigma < 0.1$ and $n > 1/\sqrt{\sigma}$, we have $s \approx 1$ $(\log_{10} s \approx 0)$ (Fig. 1: θ over $\log_{10} n - \log_{10} s$ plane; Fig. 2: dashed contours for $\theta = 0.1, 0.9$). This is a consequence of the transition region being sharp and centered at $s \approx 1$ for small σ. Since for the Residential-Commercial equilibrium we observe $\theta = 0.2$, which is within the transition region, we may assert that $s \approx 1$, provided σ is sufficiently small (see below).

For small system size n, θ is dependent on n as well as on s (Fig. 1 and Fig. 2). Thus the θ-dependent Residential-Commercial equilibrium constant, $K_{RC} = 0.25$, is a lower limit for s, the microscopic Residential-Commercial equilibrium constant. K_{RC} and s are equal at large n.

2.4.3 σ, $\mathbf{S_C}$**.** We observe that Commercial land occurs as a cluster of lattice sites, i.e., as a development site (a P-D site).[8] The origin of clustering is the cost for initiating a Commercial region (Sec. 2.3.2, Critical Size). Consequently σ must be significantly less than unity.

The dimensions of the lattice we use to model the system must be physically plausible and sufficiently fine. We set the distance between lattice points at 200 ft., corresponding to an area per lattice site in two dimensions of one acre.[33] We estimate the average size of a P-D site as 10 acres,[34] which corresponds to a cluster of size S_C = 10 lattice sites. In one dimension, $\sigma \approx (S_c)^{-2}$, and we obtain $\sigma \approx 0.01$. This estimate for σ validates the assertion made above that $s \approx 1$, which was conditional upon a small value for σ.

2.4.4 **n**, ν**.** The system size n is arbitrary as long as it is large, i.e., larger than a P-D cluster, $n > S_C \approx 1/\sqrt{\sigma}$ (Fig. 1). We set $n =$ 640, corresponding in two dimensions to one section of land. The total number of Commercial sites, $n_C = 0.2 \cdot 640 = 128$, corresponds to a border ca. 260 ft. wide along the section boundaries. The number of P-D-site clusters in a system of size $n = 640$ is $\nu = n_c/S_C = 12.8$. Metropolitan Tucson is about 1000 sections in area. The City can be treated as an ensemble of 1000 systems of the type described here.

3. Results and Discussion

We have obtained in Sec. 2.4, a self-consistent and physically plausible set of parameter estimates that describe the Residential-Commercial equilibrium of a typical urban system: $s \approx 1$; $\sigma \approx 0.01$; $n \approx 640$.

This set of values corresponds to a single point (the state point or system point) located at $\log_{10} n \approx 2.8$ and $\log_{10} s \approx 0$ on the top graph of Fig. 1 ($\sigma = 0.01$) or on the plots of Fig. 2 and 3 (calculated for $\sigma = 0.01$).

We now analyze this system within the framework of the Ising model.

3.1 Equal cost of Residential and Commercial property; The Residential-Commercial Equilibrium.

For the above set of parameter values, the microscopic equilibrium constant, s, for conversion of an interior Residential-state site to an interior Commercial-state site, is close to unity. Since the system size n is large, a near-unit value for the equilibrium constant implies that the system-wide average price for an acre of developed commercial property is closely equal to that for developed residential property.

This result, equality of price for Residential and Commercial, is expected. Market forces drive the system toward this equality: if the equilibrium is displaced from unity and commercial land has significantly greater value ($s > 1$), then speculators will be rewarded for converting to commercial use parcels of residential land adjacent to a commercial P-D site, until some factor dependent on the fraction of all sites that are Commercial sites, such as an overabundance of commercial land driving down its value, acts to restore the equilibrium value of $s \approx 1$.

From Eq. 1b, $s = e^{i_{CC}}$. The value of s is determined by the values of variables of the right-hand-side of Eq. 1a for i_{CC}, which are: the interaction of a Commercial site with its Commercial neighbors ($\delta \$_{CC}$) compared with interaction of a Residential site with its Residential neighbors ($\delta \$_{RR}$), and the profit per Commercial site ($\$P$). The profit per Commercial site is likely to fluctuate and to respond quickly and strongly to changes in conditions, accounting both for any transient displacement of the system from equilibrium and also for subsequent restoration of the equilibrium.

3.2 Importance of the Initiation Parameter, σ.

Eq. 2 states that the initiation parameter σ is determined by the cost of creating a Residential-Commercial interface. Because this cost can be large, σ can be small, as it is for the set of parameter values estimated above ($\sigma \approx 0.01$).

A value of $\sigma < 1$ is at the heart of understanding the Residential-Commercial system. As noted above, the value of s controls one important aspect of the Residential-Commercial equilibrium, the fraction of sites that are Residential or Commercial. The value of σ, however, controls perhaps more interesting and certainly more fundamental characteristics: how the system is structured (clusters); how it varies with change in s (explosive growth); the nature of catastrophes.

Unlike s, σ should vary slowly, with a characteristic time of several election cycles.

3.2.1 Clusters. As noted in section 2.3.2, Critical Size, if there is a non-zero cost of initiation, associated with $\sigma < 1$, then there must be contiguous Commercial sites sufficient in number $n \geq n_{crit}$ to collectively balance the initiation cost through the n small favorable contributions from transfer of sites from a Residential to a Commercial region. Tucson like most cities in the West of this country, shows local (on the section scale) segregation of Residential and Commercial sites, i.e., sites of each type are clustered, consistent with the estimated value of $\sigma \approx 0.01$. Not all areas of Metropolitan Tucson fit this picture of relatively small clusters of Residential and Commercial. Some areas are homogeneous on the section scale (clusters of 640 sites or more): industrial areas; suburban residential areas, such as the Foothills, without Commercial over several-mile distances; undeveloped areas.

Clustering is not a necessary characteristic of the structure of a city. Mature, densely populated cities, many on the East Coast and some on the West Coast, have large areas of contiguous sections with no clustering, where Residential and Commercial interleave both horizontally and vertically. Outside the city center, however, even in such cities, clustering is found.

3.2.2 Explosive growth. Another consequence of their being an initiation cost for a Commercial region is explosive growth of commercial regions in response to only a small increase in the equilibrium constant (the sharpness of the θ-log s response). Fig. 1 shows that a change in s from 0.8 to 1.3 switches the character of the system from 90-10 Residential-Commercial to its opposite, 90-10 Commercial-Residential. Only a small change in the value of Commercial relative to Residential property has a disproportionate, nonlinear effect on the fraction of Commercial sites.

The sharpness of the θ-log s response increases with smaller σ (greater cost of initiation), the sharpness becoming infinite as σ approaches zero.

Sharpness as a special characteristic vanishes for $\sigma = 1$, where sites do not cluster but interleave freely. For a $\sigma = 1$ system, explosive growth, dependent upon a small value of σ, is not possible.

3.3 Catastrophes.

We consider two possible catastrophes for a Residential-Commercial system:[35] Greening of the City; and Malling of it. Both would be a consequence of the the sharpness of the transition, i.e., the sensitivity of the fraction Commercial, θ, to change in the relative value of Commercial property, s. If the value of Commercial property were to be driven to 1/10 the value of Residential property ($\log_{10} s \approx -1$), there would be essentially no Commercial remaining in the system (Fig. 1), i.e., the City would have been Greened. For a 10-fold greater value of Commercial compared to Residential, the result is the opposite, and the City would have been Malled.[36] Once forced to an extreme, where there would be essentially either no Commercial or no Residential (no jobs or no homes), it is doubtful recovery could be fast.[37]

Like explosive growth, catastrophes are a consequence of a large initiation cost leading to a small value of σ and a sharp transition. Without a sharp transition, an order-of-magnitude deviation of s from its equilibrium value would not drive the system to an essentially pure Residential or pure Commercial state: compare Fig.1 for $\sigma = .01$ and 1.

What might be the origin of a large deviation in s? Most likely are external events driving the system out of equilibrium.[38] [39] Alternatively, random internal forces can drive the system. Fluctuations in the relative price of Commercial land are to be expected. Over a sufficiently long period of time,[40] the system will undergo fluctuations much larger than the typical (RMS) deviation and so large as to be catastrophic.

3.4 Optimization and the New Urbanism.

Can we optimize a Residential-Commercial equilibrium system? It should not be difficult. Three variable parameters determine the state of the system: s, σ, and n. The equilibrium constant s is restricted to be near unity by market forces (Sec. 3.1). Since the system size n is large relative to n_{crit} (Sec. 2.4.4), the transition is insensitive to change in n (Sec. 2.3.1). Thus only σ can be adjusted.

Since only σ can be adjusted, by optimization we mean minimization of the cost of creation of a Residential-Commercial interface, the initiation cost. Since the cost owes to unfavorable interaction between developer and neighborhoods adjacent to a Commercial site, it should be subject to reduction through eliminating this conflict. Reduction of the initiation cost to zero results in $\sigma = 1$, in which case there would be free interleaving of Residential and Commercial, without the clustering found for values of $\sigma < 1$, and with developer and neighbor as happy living together as separately.

One can speculate that as a city matures and develops large areas of high density Residential and Commercial, the natural evolution is toward the $\sigma = 1$ state. As one would expect for a mature city, the $\sigma = 1$ state does not show explosive growth (Sec. 3.2).

Systems in this state ($\sigma = 1$) with mixed Residential-Commercial and no clustering are found (Sec. 3.2.1), although not typically in the inland West of this country, even where as in Tucson allowed by zoning.

The paradigm of the New Urbanism corresponds to the $\sigma = 1$ state (mixed Residential-Commercial; no clustering). We may conclude that this paradigm is inapplicable for Tucson, at least for the foreseeable future and likely not until our City has evolved for several more generations and the immense area of low-cost surrounding land is no longer available.[41]

3.5 Phase Transitions.

Among the characteristics of phase transitions are: (a) emergence of a new structure, with the measure of its extent being an order parameter, here θ, the fraction of the system in the Commercial state; (b) nonlinear response to change in a system parameter, reflected in explosive growth of the new structure within the transition region; (c) cooperative interaction between elements, resulting in the above explosive growth, and also in clustering of elements of the same state and segregation of the states into separate macroscopic phases; (d) a correlation length, a characteristic size for clusters, that varies strongly within the transition region; (e) universality reflected in power laws, for example, exponential dependence of the correlation length or of the order parameter on a system variable.[42]

The last of the above list, universality, explains why a phase-transition model developed for inanimate ferromagnetic material can help us understand the city, a product of human enterprise and of far greater complexity than a magnet. The interacting elements (humans, amino acids, molecules, atoms, spins) may differ greatly in their nature, but whatever their nature, interactions lead to clusters, and associated with clusters, there is a correlation length. The basis of universality is the central importance of the correlation length, which for a system near a phase transition is the only relevant measure of length. Properties of a system undergoing a phase transition scale on the correlation length. As the view of the system is progressively coarse-grained, more long-range, the specific nature of the substances comprising the system no longer enters into the analysis. Thus one expects the formalism of phase transitions to apply equally for cities and ferromagnets.[43] [45] [46]

The Residential-Commercial equilibrium displays self-organization: (a) the state point characteristic of a city is firmly constrained by market forces to the center of the transition region ($s \approx 1$); (b) there is a characteristic cluster size, determined by σ, the initiation cost.

Complexity and complex systems are faddishly discussed in science-related news. Among the properties most closely associated with complexity are explosive growth and nonlinearity, emergence, and self-organization. It should be no surprise that the Residential-Commercial equilibrium system, a product of human enterprise, displays these diagnostic signs of complexity.

3.6 Temperature.

The two interaction parameters of the Ising model as formulated by Zimm and Bragg[17], s and σ, are defined by Eq. 1b and 2a, which are exponentials in the energy (or for our purposes, the dollar value). The exponents must be unitless.

In the energy language, the exponent is of the form $i_{CC} = -\beta E_{CC}$, where E_{CC} is the energy difference between the Commercial and Residential states and the factor β scales the energy according to its units. β is a reciprocal temperature, proportional to $1/T$, where T is the absolute temperature of the system.

As implied in Sec. 2.1.2 (footnote 22), there must be a similar scaling factor for the dollar value, which we can call $\beta_\$$. We can evaluate $\beta_\$$ by use of Eq. 2c.[47]

By analogy with β, it is plausible that $\beta_\$$ is proportional to $1/T_\$$, where $T_\$$ is a societal temperature that has a role comparable to the absolute temperature in energy language.[48]

3.7 Concluding Summary.

We have shown that an analogy with phase transitions leads to understanding of the basic principles of the Residential-Commercial equilibrium, and thus of its equivalent, the D-N problem.

We have answered the four questions raised in Sec. 1, Introduction: Fig. 1 shows the dependence of the fraction Commercial sites, θ, on system parameters; the clustering of sites has its origin in the initiation cost, σ; catastrophes would be associated with a large fluctuation in the microscopic Residential-Commercial equilibrium constant, s; optimization of an immature Residential-Commercial system is equivalent to imposing the New Urbanism model.

Can the D-N problem be solved? Yes, but only for mature cities that have evolved to fit the New Urbanism paradigm. Those who live in the immature West are condemned to live also with the D-N conflict. But more cheaply.[49]

We suggest several directions for future research. (a) *Evolution of a Residential-Commercial system with time*. We have implied this evolution, in our discussions of fluctuations in s and of the maturing of a city. A rigorous approach would require construction of a time-dependent probability density function. (b) *The entropy of a Residential-Commercial system*. Because simplification of a high-dimensional system by averaging, or projection, introduces entropy, there must be an entropic component for the energy (dollar value) terms of Eq. 1 and 2. (c) *Novel aproaches to resolution of the D-N problem*, e.g., through political and social realignment.[50]

4. Footnotes and References

1. [Editorial note: J. A. Rupley, 10/31/02.]
   I found this manuscript on my doorstep, wrapped in a blue blanket, to which was pinned the following unsigned message, "Please take care of my child... I have had to leave Tucson unexpectedly for a country with no Reciprocal Extradition Agreement." The author's name given on the manuscript is obviously a pseudonym. However, one can deduce from the subject of the manuscript that the author is involved with planning and zoning, and from the apparent flight from our City that the author is a lawyer or developer. For several months I have attended, as an observer, meetings of a subcommittee of the Planning and Zoning Commission charged with review of the City's Big Box Ordinance, and it is possible, even likely, that the author, having met me there, saw fit to entrust me with his or her creation. Wishing to avoid legal action, I will not speculate more on the author's identity. A draft cover letter indicated that the author wished the manuscript to be submitted to a specific journal, the Zeitschrift für Berüchtigtwissenschaft. I have chosen not to honor this implied request. Indeed, I have not been able to verify the existence of the so-apparently-well-named Zeitschrift.

   The author may have been under considerable stress when creating this manuscript: the footnotes referenced on the first three pages suggest that the project was begun as a vehicle for the author to vent discontent with an inimical environment. However, the initial cynical and acerbic tone vanishes entirely once the author has described the model. Was the author drawn into his own work at this point? Did the author decide to bury himself in this application of the theory of phase transitions, in order to escape briefly from a hostile world? Did the author come to believe that the Ising-model picture of the D-N problem was not just an amusement, but was essentially correct in isolating significant features of the problem? This last possibility, the author's belief in the correctness of the application, is supported by the perhaps tiresome length of the manuscript. The reader who is about to judge the author's work, is reminded of the deranged state of a person poised to flee our lovely City.

2. The D-N problem is isomorphic to the CAVE problem.[3]

3. Citizens Against Virtually Everything.

4. (a) Urban Political space is separable into the manifold of the Developer-Neighbor (D-N) problem and its orthogonal complement, the Developer-Politician (D-P) advantage.

   (b) The D-P advantage, a union of the Developer and Politician subspaces, is not an advantage but in fact a problem, in the view of neighbors and citizens at large. In any case, the Developer-Politician advantage is isomorphic to the Greed-and-Corruption problem.

   (c) The Neighbor-Politician (N-P) problem is equal to the D-N problem. Thus the CAVE problem is isomorphic to not only the D-N problem[2] but also the N-P problem, as is frequently remarked upon, particularly during election years, by elements of the D and P subspaces.[5]

   (d) N-P problems take a very long time to solve, which one speculates has led to denoting a class of problems as NP-hard.

   (e) The Urban Political function space is complex and is rife with imaginary time operators, which should surprise no one.

5. If you have a problem visualizing the Urban Political space and its subspaces, be comforted, I too have this problem.

6. The D-N problem is widely recognized as central to the planning of cities, the management of population growth, and the loss of local elections.

7. E. Ising, *Z. Physik* **31,** 253 (1925).
8. In Planner-Developer language, a site is a land area consisting of a lot or contiguous lots, designated for development as a single entity. For the present discussion, a site, i.e., a lattice site, corresponds closely to a lot or fraction of a lot. For clarity, "P-D site" denotes a development site; "site" used without qualifier means a lattice site. A P-D site is a cluster of lattice sites.[9]
9. We assume that P-D clusters are statistically independent.
10. We assume that all land can be assigned objectively as being either Residential state or Commercial state, e.g.: industrial use – Commercial state; ranch use – Residential state. Unused land is Residential state, without regard to its zoning or whether it is speculatively intended for commercial use.
11. In previous, unsuccessful treatments, these states have been denoted, respectively, paved (P) and saved (S), or profit (*$*) and loss (()).
12. The extensive footnotes are intended to comfort those who might need solace when faced with distinguishing the trace from an eigenvalue, or even a logarithm from an exponential.
13. If you disregard theory, please don't feel bad — nearly everyone seems to disregard theory, which is fortunate for theorists, who can be blunderers and yet be thought of (and think of themselves) as the top of the food chain. In any event, if theory bores you or you have no use for it, please pass over this section and go on to Results.
14. H. A. Kramers and G. H. Wannier, *Phys. Rev.* **60,** 252 (1941).
15. M. Toda, R. Kubo, and N. Saito, *Statistical Physics: 1. Equilibrium Statistical Mechanics,* 2nd edition, Springer-Verlag, New York, 1992.
16. Despite recent advances in nanotechnology, we ignore quantum effects.
17. B. H. Zimm and J. K. Bragg, *J. Chem. Phys.* **31,** 526 (1959).
18. For *our* convenience, of course. The sins of Pride, Greed, Envy, Wrath, Lust, and Gluttony may not be among ours,[19] but Sloth is, and we happen to have had already prepared in our lecture notes a tutorial on the Zimm-Bragg theory of the coil-helix transition. [For a ridiculously small honorarium, we will deliver a lecture on this subject that is deadly boring and suitable for terminating your salon or soirée by driving late-staying guests away at a moment of your choosing; please contact the Lethe Agency.]
19. If you believe that, I have a London bridge on a beautiful site in the Arizona desert, and it's for sale at a bargain price ("una ganga", as some have said).
20. You well may wonder how I can call this section "Theory", when it merely describes the mindless application of someone else's theory (Zimm and Bragg's[17]) to a congruent problem. In defense, I rush to say that referees and editors virtually insist, to be in line with tradition, that there be a section titled "Theory" in a communication of this kind. Furthermore, Zimm and Bragg did somewhat the same thing, in their application of Kramers and Wannier's[14] theory to the coil—helix transition. So also did Kramers and Wannier, who used what is now obvious mathematics to solve Ising's model,[7] which ferromagnetism afficionados had been batting around already for more than a decade. But you get the idea, and in any event, the history of science now appears to interest few and to be ignored by nearly all.
21. The electric bill demonstrates the proportionality of energy and dollars. Furthermore, as is well known, financial power is proportional to the rate of expenditure of dollars, which parallels electric power being the rate of expenditure of energy and gives independent support for the proportionality of energy and dollars.

22. Measured in unitless scaled dollars.

23. If you have difficulty visualizing the relationships of this paragraph, the following picture may help: Consider a standard quantity[24] of a gas of free Residential sites (free in the sense of independent — nothing is free dollar-wise, of course); let this gas, R(g), condense to form a Residential region, i.e., a Residential neighborhood, R(l), with the establishment of Residential-Residential interactions and with a concomitant change in energy,

$$\delta E_{RR} = \delta \$_{RR} = \$_{R(l)} - \$_{R(g)} \quad \text{(Eq. A1a)}$$

We now transform an equal quantity of gaseous Residential sites into gaseous Commercial sites, noting that a profit may be derived from Commercial but not Residential sites,

$$\delta \$_{R(g)\rightarrow C(g)} = \$_{C(g)} - \$_{R(g)} = \$(Profit/C\text{-}site) = \$P \quad \text{(Eq. A1b)}$$

and condense the Commercial-site gas to form a Commercial development, with

$$\delta E_{CC} = \delta \$_{CC} = \$_{C(l)} - \$_{C(g)} \quad \text{(Eq. A1c)}$$

By definition, for R(l) as the reference state, the Commercial-site – Commercial-site energy

$$i_{CC} = \$_{C(l)} - \$_{R(l)} \quad \text{(Eq. A1d)}$$

$$= \$(Profit/C\text{-}site) + (\delta \$_{CC} - \delta \$_{RR}) \quad \text{(Eq. A1e)}$$

that is, i_{CC} is the profit per Commercial site plus the differential gain (loss) from establishing the interactions of a Commercial site inside a Commercial region versus those of a Residential site inside a Residential region.

If one still remains confused, pass on to the Results section, since the remainder of the Theory section will confuse even more.

24. The standard quantity of a substance typically is a mole, Avogadro's number, $N_0 = 6x10^{23}$ particles. This would be rather a lot of one-acre Residential sites, even hypothetical ones. A more reasonable choice for a standard quantity would be a femtomole, corresponding to a land area approximately 1000 miles square, about 5x the territory of the four "Four Corners" states (AZ, CO, NM, UT). Neo-Malthusian enviro-futurist Greens among us would view (with horror!) an area of this size as possibly being "Mall-able" and certainly open to being zoned immediately for Residential use. Others, mindful of world demographics and the 2050 population downturn, might continue to consider a femtomole of Residential sites as somewhat excessive.

25. Mitigation of opposition of Residential neighbors to adjacent Commercial, fees (City, agents, lawyers, illegal-petition passers), interest on loans accruing as a result of bureaucratic delay (whether by intent or by incompetence), payments for politician and bureaucrat maintenance, etc.

26. It also makes physical sense: mixed use (Residential sites within a Commercial region) has been observed to be profitable.

27. A beautiful expression of the significance of the partition function, or more precisely, of averaging in statistical mechanics, has been given by Feynman.[28]

28. R. P. Feynman, *Statistical Mechanics: A Set of Lectures,* Notes taken by R. Kikuchi and H. A. Feiveson, Edited by J. Shaham, Addison-Wesley, New York, 1972.

29. The transfer matrix is an operator that propagates the vector of the statistical weights of the joint configurations of the system to the next site along the latticen chain.[15][17]

30. Zimm and Bragg obtain Z for a linear one-dimensional lattice (not wrapped), with the first three sites of the n-site chain fixed in the Residential state:

$$Z = \frac{\lambda_0^{n-2}(\lambda_0 - s) + \lambda_1^{n-2}(s - \lambda_1)}{(\lambda_0 - \lambda_1)} \qquad \text{(Eq. A2)}$$

The eigenvalues $\lambda_0 > \lambda_1$ are the same as given by Eq. 5. This expression for Z is closely similar to that of Eq. 3, and we use it for convenience, as noted.[18]

31. This development of an expression for n_{crit} is heuristic. It is intended to make plausible, through a simple argument, the clustering of sites of each type, Residential and Commercial. S_C, the average size of a cluster, defined in Sec. 2.3.4, is a more theoretically-proper measure of clustering. However, S_C depends on ν, defined by Eq. 10, which is not as obvious in its physical meaning as Eq. 9. Note that Eq. 9 fails for $s \leq 1$.

32. The fraction of the full cash value that is Commercial was 17.88 percent for Pima County in 2001 (data from Pima County Assessor's Office).

33. We move freely and without apology between 2-D and 1-D pictures. Most readers should be familiar with 2-D representations of a city (known to some as maps). We use this familiarity to explain aspects of the 1-D model. If one requires rigor, a 2-D lattice can be mapped onto a 1-D chain, e.g., by projection.

34. Based only on personal observation and anecdotal information. However, a crude estimate (order of magnitude) is sufficient for our intent, to show the general features of the problem.

35. There are other possible catastrophes. Some would consider it a catastrophe were a Wal*Mart to be introduced into a Residential-Commercial system.

36. This is the night-sweat fear of the above-noted neo-Malthusian enviro-futurist Green.

37. Thus our citizenry would face either a bucolic life barren of civilizing contact or the bitter life of the Commercial wasteland.

38. Consider the effect of a massive inflow of wealthy Californians, seeking refuge from the debacle impending for that state, who could drive the price of Residential property in Tucson higher than whence they came.

39. Even without the Californians, it is possible that Tucson politicians might run amok California-style and impose surreal disadvantages upon Commercial land.

40. Possibly a very, very, very long time.

41. Designer fads, whether urban planning or clothes fashion, typically are generated on one of our Coasts, or with even less appropriateness, in Europe. The New Urbanism, like fashionable clothing, does not mix well with one-ton pickups.

42. There are difficulties associated with a 1-D model: no true phase transition (segregation of phases) except at infinitely small σ; no singularities in the power laws, e.g., no infinite correlation length at the critical point.

43. This paragraph is an attempt to discuss universality and scaling with the limited intent of explaining how phase transitions for different substrates can be fundamentally similar, and without any intent to consider singularities at the critical point, critical exponents, dimensionality, and other aspects of critical phenomena and their analysis. For a good treatment of phase transitions, see the text by Kubo,[15] or the excellent treatment in the text by Huang.[44]

44. K, Huang, *Statistical Mechanics*, 2nd Edition, John Wiley, New York, 1987.

45. Indeed, phase transitions can be thought of as endemic in the natural and social sciences. Wherever there is explosive growth and emergent structure, think, Phase Transition! Such models have been applied widely: ferromagnetism, protein folding, carriage of oxygen by hemoglobin, membrane phenomena, epidemics, gelation, liquid helium, formation of galaxies, etc.

46. One might have difficulty persuading the officials of a large city to allow a series of experiments in which system variables (s and σ) are controlled over a sufficiently wide range of values to explore the transition region and determine power-law behavior.

47. From Eq. 2c, for $|i_{RC}| \gg |i_{CC}|$,

    $$\beta_\$ = -\log\sigma / (cost\ of\ creation\ of\ R - C\ interface),$$

    where the cost is in unscaled dollars. We make the following estimates: $\log\sigma \approx -2$ (Sec. 2.4.3); an interface cost equal to 10 percent of the value of developed residential property in the interface; an average over Metropolitan Tucson of \$400,000 per developed residential acre (e.g., 4 residences/acre on 1/4-acre lots); two Residential sites in the interface of a Commercial site within a Residential region. Thus

    $$\beta_\$ \approx -(-2)/(2 * 0.1 * \$400,000) = 1/\$40,000.$$

48. $\beta_\$$ determines what a dollar buys for you in the way of developed Residential or Commercial property. For example, because Tucson clearly has a higher societal temperature T_s than Benson, one buys better, and so had better buy, in Benson.

49. In the New Urbanism model, $\sigma = 1$ and there is no cost of mitigating or otherwise overcoming developer-neighbor conflict. One might believe that land, both Commercial and Residential, would be relatively less costly in a city that fit the New Urbanism model. This naive expectation is of course not the case. Because the New Urbanism model requires a mature city, the multitude of bureaucrats, lawyers, politicians, agents, and others of similar ilk have available the refinements made by previous generations in the methods for extracting ever greater blocks of money from both developer and residents. No moral here, just depressing fact.

50. Both developer and neighbors suffer costs associated with participation of the zoning bureaucracy and politicians in the D-N problem. These costs are a transfer of funds from citizens (developers and neighbors) to politicians, lawyers, bureaucrats, etc., over and above whatever costs are required for actual mitigation at the development site. We suggest seeking Foundation grants to explore the formation of a non-profit Institute, created and administered jointly by developers and neighbors, with the mission of compulsory mediation of D-N conflicts. Participation of an entity of this kind could reduce or eliminate the current substantial overhead costs. Political action under the auspices of the Institute could eliminate planning departments (they do no planning anyway), reduce graft, bribery and other aspects of politician maintenance, shrink the size of local government, reduce the number of lawyers, foster a happy citizenry, etc. Developers would support such a plan — they will vote for a yellow dog if it would improve the bottom line. Neighbors will vote however they are told — but in such a way as never to agree.

5. Figures and Legends

Figure 1. Fraction Commercial-sites, θ, over $\log_{10} n - \log_{10} s$ plane: top, for $\sigma = 0.01$; middle, for $\sigma = 0.1$; bottom, for $\sigma = 1$. n is the system size (total number of sites, $n = n_C + n_R$); s is the microscopic Residential–Commercial equilibrium constant, with $\log s = i_{CC}$. Values of θ were calculated by use of Eq. 7b.

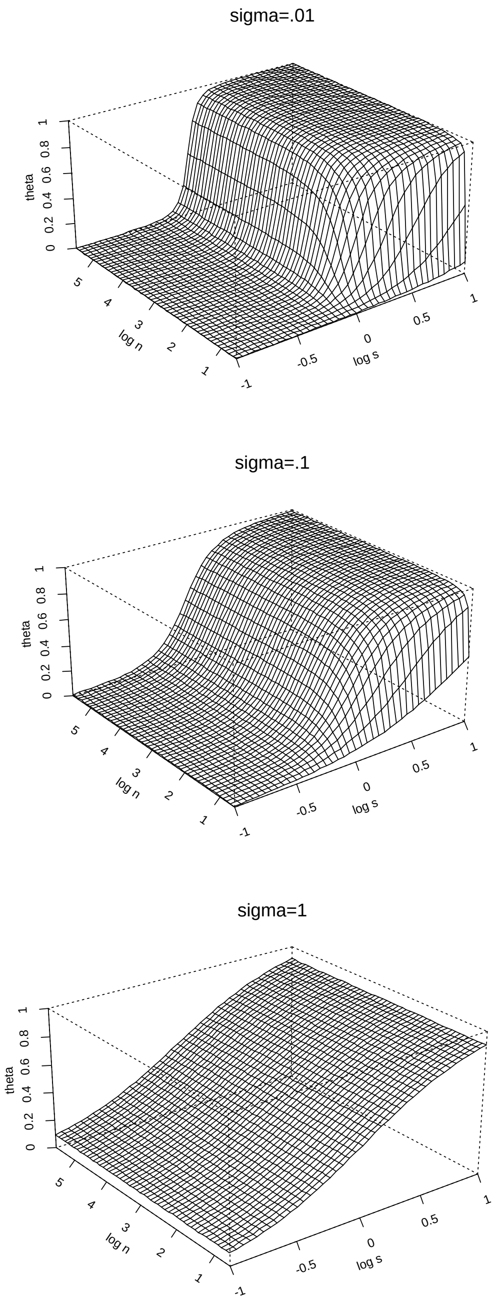

Figure 2. $\log_{10}\nu$ over $\log_{10} n$– $\log_{10} s$ plane, for $\sigma = 0.01$. ν is the number of Commercial regions, i.e., clusters of Commercial sites, in a system of size n. Solid lines, contours for designated values of $\log_{10}\nu$; dashed lines, contours for $\theta = 0.1$ and 0.9, for lower and upper range, respectively, of the Residential-Commercial transition region of Fig. 1 top graph (for $\sigma = 0.01$). Values of ν were calculated by use of Eq. 10.

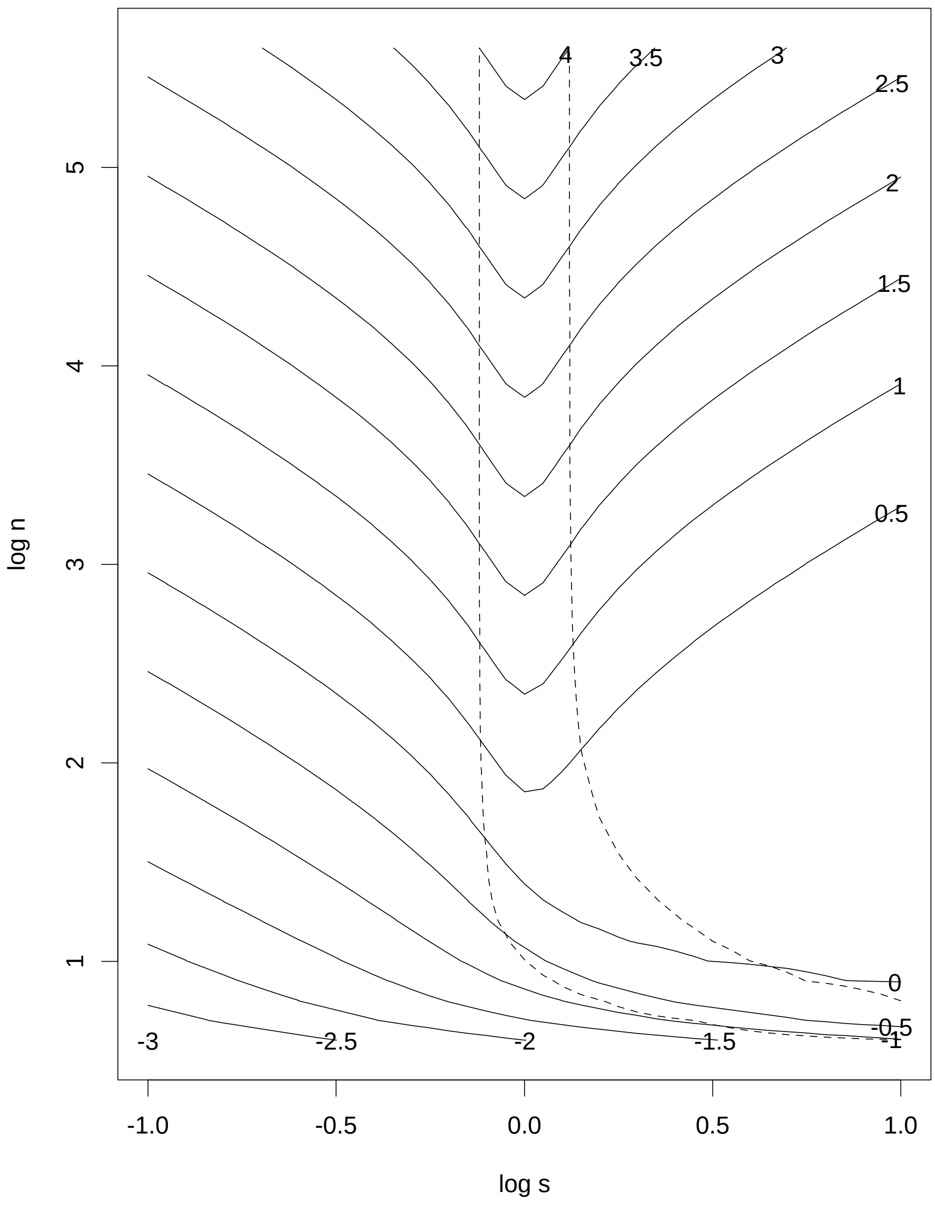

Figure 3. $\log_{10} S_C$ over $\log_{10} n - \log_{10} s$ plane, for $\sigma = 0.01$. S_C is the average size of a Commercial region. Solid lines, contours for designated values of $\log_{10} S_C$; dashed lines, contours for $\theta = 0.1$ and 0.9 (bounds of the Residential-Commercial transition region for $\sigma = 0.01$). Values of S_C were calculated by use of Eq. 10 and 12a.